

# Momentum and Angular Momentum in the Expanding Universe

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## Abstract

A new approach has been used to evaluate the momentum and angular momentum of the isotropic and homogeneous cosmological models. It is shown that the results obtained for momentum exactly coincide with those already available in the literature. However, the angular momentum expression coincides only for the closed Friedmann model.

## 1 Introduction

There have been various attempts [1-4] to evaluate energy, momentum and angular momentum in the expanding universe. Rosen [1] and Cooperstock [2] used the Einstein pseudotensor with Cartesian coordinates to calculate the total energy of a closed Friedmann Robertson Walker (FRW) universe.

Janusz Garecki [3] used the Pirani [5-7] and Komar solutions [6-8, 9 (chap.11)] to evaluate the energy and other quantities for the isotropic and homogeneous cosmological models which exist in General Relativity (GR). The line we wish to follow here uses the Newtonian force concept adapted to Relativity [10,11]. The idea of re-introducing the Newtonian gravitational force into the theory of GR [10] arose in an attempt to deal with the following problem: Gravitation, being non-linear, should dominate over the Coulomb interaction at some, sufficiently small, scale. At what scale would

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it occur? Whereas this question is perfectly valid in pre-relativistic terms it becomes meaningless in GR. The reason is that gravitation is expressed in purely geometric terms while electromagnetism is not. Thus, in Relativity, gravitation possesses a very different status than the other forces of Nature. Our physical intuition for the other interactions, nevertheless, rests on the concept of forces. To deal with gravity and other forces together, we must either express the other forces geometrically, as in the Kaluza-Klein theories [12], or express gravitation in the same terms as the other forces. We will follow the latter alternative as the simpler program to implement.

The plan of the paper is as follows. In the next section we shall briefly review the essential points of the extended pseudo Newtonian ( $e\psi N$ )-formalism for the purpose of application. In section three we apply the formalism to the Gödel universe and evaluate its force, momentum and angular momentum. In the next section we shall use this formalism to calculate force and momentum for isotropic and homogeneous cosmological models. In section five we shall evaluate the angular momentum for the expanding models. Finally, in the last section we summarise the results.

## 2 The $e\psi N$ -Formalism

The basis of the formalism is the observation that the tidal force, which is operationally determinable, can be related to the curvature tensor by

$$F_T^\mu = m R_{\nu\rho\pi}^\mu t^\nu l^\rho t^\pi, \quad (\mu, \nu, \rho, \pi = 0, 1, 2, 3), \quad (1)$$

where  $m$  is the mass of a test particle,  $t^\mu = f(x)\delta_0^\mu$ ,  $f(x) = (g_{00})^{-1/2}$  and  $l^\mu$  is the separation vector.  $l^\mu$  can be determined by the requirement that the tidal force have maximum magnitude in the direction of the separation vector. Choosing a gauge in which  $g_{0i} = 0$  (similar to the synchronous coordinate system [13]) in a coordinate basis. We further use Riemann normal coordinates (RNCs) for the spatial direction, but not for the temporal direction. The reason for this difference is that both ends of the accelerometer are spatially free, i.e. both move and do not stay attached to any spatial point. However, there is a “memory” of the initial time built into the accelerometer in that the zero position is fixed then. Any change is registered that way. Thus “time” behaves very differently from “space”.

The relativistic analogue of the Newtonian gravitational force called the  $\psi N$  gravitational force, is defined as the quantity whose directional deriva-

tive along the accelerometer, placed along the principal direction, gives the extremised tidal force and which is zero in the Minkowski space. Thus the  $e\psi N$  force,  $F_\mu$ , satisfies the equation

$$F_T^{*\mu} = l^\nu F_{;\nu}^\mu, \quad (2)$$

where  $F_T^{*\mu}$  is the extremal tidal force corresponding to the maximum magnitude reading on the dial. Notice that  $F_T^{*0} = 0$  does not imply that  $F^0 = 0$ . With the appropriate gauge choice and using RNCs spatially, Eq.(2) can be written in a space and time break up as

$$l^i(F_{,i}^0 + \Gamma_{ij}^0 F^j) = 0, \quad (3)$$

$$l^j(F_{,j}^i + \Gamma_{0j}^i F^0) = F_T^{*i} \quad (4)$$

A simultaneous solution of the above equations can be found by taking the ansatz [14]

$$F_0 = -m \left[ \{\ln(Af)\}_{,0} + g^{ik} g_{jk,0} g^{jl} g_{il,0} / 4A \right], \quad F_i = m(\ln f)_{,i} \quad (5)$$

where  $A = (\ln \sqrt{-g})_{,0}$ ,  $g = \det(g_{ij})$ . This force formula depends on the choice of frame, which is not uniquely fixed.

The new feature of the  $e\psi N$  force is its zero component. In special relativistic terms, which are relevant for discussing forces in a Minkowski space, the zero component of the four-vector force corresponds to a proper rate of change of energy of the test particle. Further, we know that in general an accelerated particle either radiates or absorbs energy according as  $\frac{dE}{dt}$  is less or greater than zero. Thus  $F_0$ , here, should also correspond to energy absorption or emission by the background spacetime. Infact we could have separately anticipated that there should be energy non-conservation as there is no timelike isometry. In that sense  $F_0$  gives a measure of the extent to which the spacetime lacks isometry.

Another way of interpreting  $F_0$  is that it gives measure of the change of the "gravitational potential energy" in the spacetime. In classical terms, neglecting this component of the  $e\psi N$  force would lead to erroneous conclusions regarding the "energy content" of the gravitational field. Cotrariwise, including it enables us to revert to classical cocepts while dealing with a general relativistically valid treatment. It can be hoped that this way of looking at energy in relativity might provide a pointer to the solution of the problem of definition of mass and enrgy in GR.

The spatial component of the  $e\psi N$  force  $F_i$  is the generalisation of the force which gives the usual Newtonian force for the Schwarzschild metric and a " $\frac{Q^2}{r^3}$ " correction to it in the Riesen-Nordstrom metric [10]. The  $\psi N$  force may be regarded as the "Newtonian fiction" which "explains" the same motion (geodesic) as the "Einsteinian reality" of the curved spacetime does. We can, thus, translate back to Newtonian terms and concepts where our intuition may be able to lead us to ask, and answer, questions that may not have occurred to us in relativistic terms. Notice that  $F_i 0$  does not mean deviation from geodesic motion.

The quantity whose proper time derivative is for the test particle. Thus the momentum four-vector,  $p_\mu$ , is

$$p_\mu = \int F_\mu dt. \quad (6)$$

The spatial components of this vector give the momentum imparted to test particles as defined in the preferred frame (in which  $g_{0i} = 0$ ).

Consider a "test rod" (the 1-dimensional extension of a test particle) of length  $\lambda$  in the preferred reference frame. The spin vector is given [15] by

$$S^\mu = \frac{1}{2} e^{\mu j k \nu} e_{j k l} l^\nu p_\nu, \quad (7)$$

where  $e^{\mu\nu\rho\pi}$  is the totally skew fourth rank tensor. In the preferred frame the spin vector will be proportional to  $l^i$ , so that

$$S^i = p_0 l^i. \quad (8)$$

(Here the spin can be taken to be negative if the sign of the preferred direction is reversed.) Taking the magnitude of the spin vector the angular momentum [16] imparted to the test rod is

$$s = p_0 \lambda, \quad (9)$$

where  $p_0$  is the zero component of the four-vector momentum. Hence the physical significance of the zero component of the momentum four-vector would be that it provides an expression for the spin imparted to a test rod in an arbitrary spacetime.

### 3 Application of the Formalism to the Gödel Universe Model

In 1949, Kurt Gödel gave an exact solution of Einstein's field equations in which the matter takes the form of a pressure-free perfect fluid ( $T_{ab} = \rho u_a u_b$ , where  $\rho$  is the matter density and  $u_a$  the normalised four-vector velocity). The metric can be given [17] in the form

$$ds^2 = -dt^2 + dr^2 - \frac{1}{2}e^{2ar}d\theta^2 + dz^2 - 2e^{ar}dtd\theta, \quad (10)$$

where  $a$  is a constant. The  $e\psi N$ -force, for the Gödel universe model, will become

$$F_0 = 0, \quad F_1 = ma \frac{e^{2ar}}{1 - e^{2ar}}, \quad F_2 = 0 = F_3 \quad (11)$$

The corresponding quantities  $p_0$  and  $p_i$ , will turn out to be

$$p_0 = \text{constant}, \quad p_1 = ma \frac{e^{2ar}}{1 - e^{2ar}}t + \text{constant}, \quad p_2 = \text{constant} = p_3 \quad (12)$$

Thus the spin angular momentum becomes

$$s = \text{constant} \quad (13)$$

### 4 Isotropic and Homogeneous Cosmological Models in GR

The isotropic and homogeneous cosmological models which exist in the framework of the GR give us good standard mathematical models of the real universe, called the Friedmann models. The Friedmann cosmological models are given by

$$ds^2 = dt^2 - a^2(t) \left[ d\chi^2 + \sigma^2(\chi) d\Omega^2 \right], \quad (14)$$

where  $\chi$  is the hyperspherical angle,  $\sigma(\chi)$  is  $\sinh \chi$ ,  $\chi$  or  $\sin \chi$  according as the model is open ( $k = -1$ ), flat ( $k = 0$ ), closed ( $k = 1$ ) and  $a(t)$  is the

corresponding scale factor. For matter-dominated Friedmann models,  $a(t)$  is given by

$$a(t) = a_0^{1/3} t^{2/3}, \quad (k = 0), \quad (15)$$

$$a(t) = \frac{a_0}{2}(\cosh \eta - 1), \quad t = \frac{a_0}{2}(\sinh \eta - \eta), \quad \eta \geq 0, \quad (k = -1), \quad (16)$$

$$a(t) = \frac{a_0}{2}(1 - \cos \eta), \quad t = \frac{a_0}{2}(\eta - \sin \eta), \quad 0 \leq \eta \leq 2\pi, \quad (k = 1), \quad (17)$$

where  $a_0$  is a constant. The  $e\psi N$  force, for the Friedmann models, is simply

$$F_0 = -m\ddot{a}/\dot{a}, \quad F_i = 0, \quad (18)$$

where a dot denotes differentiation with respect to  $t$ . The corresponding  $p_0$  and momentum,  $p_i$ , imparted to a test particle is

$$p_0 = -m \ln \dot{a}, \quad p_i = \text{constant}. \quad (19)$$

For a flat Friedmann model, Eq.(18) yields

$$F_0 = -m/3t, \quad F_i = 0. \quad (20)$$

Thus  $F_0$  is proportional to  $t^{-1}$  and hence goes to  $\infty$  as  $t$  approaches to 0 and it tends to 0 when  $t$  approaches to  $\infty$ . Since  $F_0$  is negative, it corresponds to the energy dissipation [14] by the background spacetime.

Consequently,  $p_0$  and  $p_i$  become

$$p_0 = \frac{1}{3}m \ln (T/t), \quad p_i = \text{constant}, \quad (21)$$

where  $T$  is a constant.

For the open Friedmann model, at the arbitrary times, the components of the  $e\psi N$  force turn out to be

$$F_0 = \frac{2m}{a_0 \sinh \eta (\cosh \eta - 1)}, \quad F_i = 0. \quad (22)$$

Hence  $F_0$  goes as  $t^{-1}$  for large  $t$  as in the case for flat Friedmann model. The corresponding  $p_0$  and  $p_i$  are

$$p_0 = m \ln \left( \frac{\cosh \eta - 1}{\sinh \eta} \right), \quad p_i = \text{constant}. \quad (23)$$

For the closed Friedmann universe, the  $e\psi N$  force becomes

$$F_0 = \frac{2m}{a_0 \sin \eta (1 - \cos \eta)}, \quad F_i = 0. \quad (24)$$

This creates a problem at the phase of maximum expansion. However, it can be resolved by redefining [14] the zero setting of the accelerometer. This can be explained as follows. The Christoffel symbol  $\Gamma_{0j}^i$  appearing in Eq.(4) is zero, for this case, when  $F_T^*$  reaches a minimum value. According to the ansatz used this gives an infinite  $e\psi N$  force at that instant. This is clearly absurd. It was verified that obtaining the general solution to Eqs.(3) and (4) does not resolve this problem. However, there is an arbitrariness in what we choose to call the “zero” of the accelerometer. There is no *a priori* reason to set it at any particular value. We can then choose to set it at zero at the phase of maximum expansion,  $\eta = \pi$ , so as to avoid the infinity in the  $e\psi N$  force. Using this resetting, the  $e\psi N$  force takes the form

$$F_0 = m \frac{4 + 3 \sin^2 \eta + 3 \cos \eta + \cos^3 \eta}{4a_0 \sin \eta (1 - \cos \eta)}, \quad F_i = 0. \quad (25)$$

This gives  $F_\mu = 0$  at the phase of maximum expansion,  $\eta = \pi$ , of the universe.

The corresponding quantities  $p_0$  and  $p_i$  are

$$p_0 = m \left[ \frac{1}{2} \ln \left| \frac{1}{1 - \cos \eta} \right| + \frac{3}{8} \cos \eta + \frac{1}{16} \sin^2 \eta + c_1 \right], \quad p_i = c_2 \quad (26)$$

where  $c_1$  and  $c_2$  are arbitrary constants. Hence, at the phase of maximum expansion, we have

$$p_0 = m \left[ \ln \left( \frac{1}{\sqrt{2}} \right) - \frac{3}{8} + c_1 \right], \quad p_i = c_2 \quad (27)$$

We see that the momentum,  $p_i$ , is constant in all the three cases of the Friedmann universe. Since  $c_1$  and  $c_2$  are arbitrary constants we can choose them as zero. This choice makes the momentum four-vector,  $p_\mu$ , zero at the phase of maximum expansion.

## 5 Angular Momentum in Isotropic and Homogeneous Cosmological Models

The angular momentum, for the Friedmann models, is

$$s = p_0 \lambda = -m\lambda \ln \dot{a} \quad (28)$$

For a flat Friedmann model, Eq.(28) gives

$$s = \frac{1}{3}m\lambda \ln (T/t) . \quad (29)$$

For the open Friedmann model, at the arbitrary times, the angular momentum will become

$$s = m\lambda \ln \left( \frac{\cosh \eta - 1}{\sinh \eta} \right) . \quad (30)$$

For the closed Friedmann universe, the angular momentum takes the form

$$s = m\lambda \left[ \frac{1}{2} \ln \left| \frac{1}{1 - \cos \eta} \right| + \frac{3}{8} \cos \eta + \frac{1}{16} \sin^2 \eta + c_1 \right] . \quad (31)$$

Hence, at the phase of maximum expansion, we have

$$s = m\lambda \left[ \ln \left( \frac{1}{\sqrt{2}} \right) - \frac{3}{8} + c_1 \right] . \quad (32)$$

Since  $c_1$  is arbitrary constant. Thus for a suitable choice of this constant we have the zero spin angular momentum at the phase of maximum expansion.

## 6 Conclusion

We have applied the  $e\psi N$ -formalism to the Gödel universe model. Since this model is stationary (non-expanding), the temporal component of the force must have to be zero which is the requirement of the procedure. Consequently, the spin angular momentum becomes constant. Also, the model is only radial coordinate dependent, thus we have the force and momentum only in the radial direction. As a result, the momentum in the other directions is constant. It is worth noticing that the results turn out as the natural requirement of the formalism for non-expanding universe.

Next, we have calculated momentum of a test particle by using  $e\psi N$ -formalism. It has been shown that this quantity turns out to be constant in the free fall rest-frame of the cosmic fluid for the isotropic and homogeneous relativistic cosmological models. Further we note that the quantity



$p_0$  vanishes for  $c_1 = \frac{3}{8} - \ln \frac{1}{\sqrt{2}}$  in a closed Friedmann universe at the phase of maximum expansion and the quantity  $p_i$  can be made zero for  $c_2 = 0$ . Thus the momentum four-vector becomes zero for this particular choice of constants at the phase of maximum expansion. We have also evaluated spin angular momentum for the Friedmann universe. This quantity becomes constant for the closed model. If we choose a particular value of the arbitrary constant  $c_1$  as above, we can have the zero spin angular momentum for the closed universe at the phase of maximum expansion. This is what one would expect from the formalism. These results coincide with the momentum and angular momentum results given by Garecki [3]. However, the angular momentum do not become zero for the flat and open Friedmann models. This seems to be a problem. To understand the angular momentum imparted to a test rod, a complete analysis is under progress [18] with reference to gravitational waves.

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